Wireless Communications
From 5G and WiFi 6 to Low Power IoT

Lecture 3: Wireless Channel
Haitham Hassanieh
Previous Lecture:

- Up Conversion & Down Conversion of Complex Signals
- Review of Gaussian Noise
- Types Digital Modulation
- QAM: Maximum Likelihood Decoding & BER vs. SNR

This Lecture:

- Pulse Shaping
- Matched Filter
- Multipath Channel
- Channel Estimation & Correction
- Narrowband vs. Wideband Channels
- Channel Equalization
How many bits does each symbol encode in 256 QAM Modulation?

A. 4 bits
B. 8 bits
C. 16 bits
D. 256 bits
Which of the following bits to symbol mapping in 4 QAM results in lower BER?

A. Mapping 1
B. Mapping 2
C. They both have equal BER.
D. Not possible to know without computing BER exactly.
Suppose the SNR per bit \((E_b/N_0)\) at the receiver is 14 dB, what modulation should we use if we want to guarantee a BER < \(10^{-2}\)?

A. BPSK
B. 16 QAM
C. 64 QAM
D. Any of them.
Wireless Communication System

Bits-to-Symbols Mapper
Modulation (Encoding)

Pulse Shaping
DAC
LPF

Mixer
BPF
PA
PLL

Symbols-to-Bits Mapper
Demodulation (Decoding)

Channel Equalization & Synchronization
ADC
LPF
BPF
LNA
PLL
Wireless Communication System

Bits -to- Symbols Mapper

Modulation (Encoding)

Bits

Pulse Shaping

DAC

LPF

Mixer

BPF

PA

PLLL

TX

10110011001

RX

10110111001

Bits -to- Symbols Mapper

Symbols-to-Bits Mapper

Channel Equalization & Synchronization

Demodulation (Decoding)
PAM: Pulse Amplitude Modulation

\[ b[n] = 10110001100110 \]

Bits-to-Symbols Mapper

\[ s[n] = -1, -3, +1, +3, -1, +3, -1, \]

\[ x(t) = \sum_{n=-\infty}^{+\infty} s[n] p(t - nT_b) \]

- \( T_b \): Symbol Time
- \( R_b = \frac{1}{T_b} \): Symbol Rate
- \( p(t) \): pulse shape
Pulse Shaping

\[ b[n]:10110001100110 \]

Bits-to-Symbols Mapper
Modulation (Encoding)

\[ s[n]:-1,-3,+1,+3,-1,+3,-1, \]

\[ x(t) = \sum_{n=-\infty}^{+\infty} s[n]p(t - nT_b) \]

- Simplest pulse shape: Rectangle

\[ p(t) = \Pi \left( \frac{t}{T_b} \right) \]
Pulse Shaping: Rectangular Pulse

\[ p(t) = \Pi \left( \frac{t}{T_b} \right) \]

\[ P(f) = T_b \text{sinc}(\pi T_b f) \]
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\[ P(f) = T_b \text{sinc}(\pi T_b f) \]

Very wide bandwidth!
Low spectral efficiency

\[ |P(f)|^2 \approx -13dB \]
Pulse Shaping: Rectangular Pulse

Sampled Values: $r[n] = -1, -3, +1, +3, -1, +1, -1 \ldots$

$\Rightarrow r[n] = s[n] \Rightarrow \text{No Inter-Symbol-Interference (ISI)}$

ISI: $r[n] = s[n] + p(T_b)s[n - 1] + p(2T_b)s[n - 2] + \ldots$

Rectangular Pulse $\Rightarrow$ No ISI
Pulse Shaping: Rectangular Pulse

\[ p(t) = \Pi \left( \frac{t}{T_b} \right) \]

\[ P(f) = T_b \text{sinc}(\pi T_b f) \]

Very wide bandwidth!
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No Inter-Symbol-Interference
Pulse Shaping: Rectangular Pulse

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Very wide bandwidth!

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No Inter-Symbol-Interference

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Pulse Shaping: Rectangular Pulse

\[ p(t) = \Pi \left( \frac{t}{T_b} \right) \]

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\[ p(t) \propto \text{sinc} \left( \frac{2\pi t}{T_b} \right) \]

\[ |P(f)|^2 \]
Pulse Shaping: Rectangular Pulse

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Pulse Shaping: Rectangular Pulse

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\[ |P(f)|^2 \]

\[ \begin{array}{c}
p(t) \\
p(t - T_b)
\end{array} \]
Pulse Shaping: Rectangular Pulse

\[ p(t) \propto \text{sinc} \left( \frac{2\pi t}{T_b} \right) \]

\[ |P(f)|^2 \]
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| \[|P(f)|^2\]
Pulse Shaping: Rectangular Pulse

\[ p(t) \propto \text{sinc}\left( \frac{2\pi t}{T_b} \right) \]

\[ |P(f)|^2 \]

\[ t \]

\[ 0 \quad T_b \quad 2T_b \quad 3T_b \quad 4T_b \quad 5T_b \]
Pulse Shaping: Rectangular Pulse

\[ p(t) \propto \text{sinc} \left( \frac{2\pi t}{T_b} \right) \]

\[ |P(f)|^2 \]

- Infinite Response \(\Rightarrow\) Impossible to realize in practice
- High ISI if sampling is not perfect
- Bandlimited in Frequency
Pulse Shaping: Raised Cosine
Pulse Shaping: Raised Cosine

\[
P(f) = \begin{cases} 
T_b & \text{if } |f| \leq \frac{1 - \alpha}{2T_b} \\
\end{cases}
\]

<table>
<thead>
<tr>
<th>( f )</th>
<th>(-\frac{1 - \alpha}{2T_b})</th>
<th>( \frac{1 - \alpha}{2T_b} )</th>
<th>( T_b )</th>
<th>( \frac{1 - \alpha}{2T_b} )</th>
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<td>( P(f) )</td>
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\[
|f| \leq \frac{1 - \alpha}{2T_b}
\]
Pulse Shaping: Raised Cosine

\[
P(f) = \begin{cases} 
  T_b & |f| \leq \frac{1 - \alpha}{2T_b} \\
  \frac{T_b}{2} \left[ 1 + \cos \left( \frac{\pi T_b}{\alpha} \left( |f| - \frac{1 - \alpha}{2T_b} \right) \right) \right] & \frac{1 - \alpha}{2T_b} \leq |f| \leq \frac{1 + \alpha}{2T_b}
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\frac{T_b}{2} \left[ 1 + \cos \left( \frac{\pi T_b}{\alpha} \left( |f| - \frac{1 - \alpha}{2T_b} \right) \right) \right] & \frac{1 - \alpha}{2T_b} \leq |f| \leq \frac{1 + \alpha}{2T_b} \\
0 & \text{otherwise}
\end{cases}
\]

\[
|P(f)|^2
\]
Pulse Shaping: Raised Cosine

\[ P(f) = \frac{1}{2} \left( 1 + \cos \left( \pi f \frac{T_b}{T_b} \right) \right) \text{ for } |f| \leq \frac{1}{2T_b} \]
Pulse Shaping: Raised Cosine

\[ P(f) = \begin{cases} 
T_b & \text{for } |f| \leq \frac{1}{4T_b} \\
\frac{T_b}{2} \left[ 1 + \cos \left( 2\pi T_b |f| - \frac{\pi}{2} \right) \right] & \frac{1}{4T_b} \leq |f| \leq \frac{3}{4T_b} \\
0 & \text{otherwise}
\end{cases} \]

\[ \alpha = 0.5 \]
Pulse Shaping: Raised Cosine

\[ P(f) = \begin{cases} 
T_b & \text{if } |f| \leq \frac{3}{8T_b} \\
\frac{T_b}{2} \left[ 1 + \cos \left( 4\pi T_b |f| - \frac{3\pi}{2} \right) \right] & \text{if } \frac{3}{8T_b} \leq |f| \leq \frac{5}{8T_b} \\
0 & \text{otherwise}
\end{cases} \]
Pulse Shaping: Raised Cosine

\[
P(f) = \begin{cases} 
  \frac{T_b}{2} [1 + \cos(\pi T_b |f|)] & |f| \leq \frac{1}{T_b} \\
  0 & \text{otherwise}
\end{cases}
\]
Pulse Shaping: Raised Cosine

\[ P(f) = \begin{cases} 
T_b & \text{if } |f| \leq \frac{1}{2T_b} \\
0 & \text{otherwise} 
\end{cases} \]

\[ \alpha = 1 \]  \[ \alpha = 0.5 \]  \[ \alpha = 0.25 \]  \[ \alpha = 0 \]
Pulse Shaping: Raised Cosine

\[ P(f) = \begin{cases} 
T_b & |f| \leq \frac{1 - \alpha}{2T_b} \\
\frac{T_b}{2} \left[ 1 + \cos \left( \frac{\pi T_b}{\alpha} \left( |f| - \frac{1 - \alpha}{2T_b} \right) \right) \right] & \frac{1 - \alpha}{2T_b} \leq |f| \leq \frac{1 + \alpha}{2T_b} \\
0 & \text{otherwise}
\end{cases} \]

- \( \alpha = 1 \)
- \( \alpha = 0.5 \)
- \( \alpha = 0.25 \)
- \( \alpha = 0 \)

\( \alpha \): Rolloff Factor

↑ \( \alpha \) : ↑ Bandwidth leakage
↓ \( \alpha \) : ↓ Bandwidth leakage
Pulse Shaping: Raised Cosine

\[ P(f) = \begin{cases} 
T_b & \text{if } |f| \leq \frac{1 - \alpha}{2T_b} \\
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0 & \text{otherwise}
\end{cases} \]

\[ p(t) = \text{sinc}(t/T_b) \frac{\cos(\pi \alpha t/T_b)}{1 - 4 \alpha^2 t^2/T_b^2} \]

\( \alpha \): Rolloff Factor

↑ \( \alpha \) : ↑ Bandwidth leakage
↓ \( \alpha \) : ↓ Bandwidth leakage
Pulse Shaping: Raised Cosine

\[ P(f) = \text{sinc}(\frac{t}{T_b}) \frac{\cos(\pi \alpha t / T_b)}{1 - 4 \alpha^2 t^2 / T_b^2} \]

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\( \alpha \): Rolloff Factor

\( \uparrow \alpha : \uparrow \) Bandwidth leakage

\( \downarrow \alpha : \downarrow \) Bandwidth leakage
Pulse Shaping: Raised Cosine

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p(t) = \text{sinc}(t/T_b) \frac{\cos(\pi \alpha t/T_b)}{1 - 4 \alpha^2 t^2/T_b^2}
\]

\(\alpha\): Rolloff Factor

\(\uparrow \alpha : \uparrow\) Bandwidth leakage

\(\downarrow \alpha : \downarrow\) Bandwidth leakage
Pulse Shaping: Raised Cosine

\[ P(f) = \frac{\cos(\pi \alpha t/T_b)}{1 - 4 \alpha^2 t^2/T_b^2} \]

\[ p(t) = \text{sinc}(t/T_b) \frac{\cos(\pi \alpha t/T_b)}{1 - 4 \alpha^2 t^2/T_b^2} \]

\( \alpha \): Rolloff Factor

\( \uparrow \alpha \): \( \uparrow \) Bandwidth leakage

\( \downarrow \alpha \): \( \downarrow \) Bandwidth leakage

\( \alpha = 0 \)
\( \alpha = 0.25 \)
\( \alpha = 0.5 \)
\( \alpha = 1 \)
Pulse Shaping: Raised Cosine

\[ P(f) = \begin{cases} \frac{1}{2T_b} & -\frac{1}{2T_b} < t < \frac{1}{2T_b} \\ 0 & \text{otherwise} \end{cases} \]

\[ p(t) = \text{sinc} \left( \frac{t}{T_b} \right) \frac{\cos(\pi \alpha t / T_b)}{1 - 4 \alpha^2 t^2 / T_b^2} \]

\( \alpha \): Rolloff Factor

\( \uparrow \alpha \): \uparrow Bandwidth leakage

\( \downarrow \alpha \): \downarrow Bandwidth leakage
Pulse Shaping: Raised Cosine

\[ p(t) = \text{sinc}(t/T_b) \frac{\cos(\pi at/T_b)}{1 - 4 \alpha^2 t^2 / T_b^2} \]

\[ P(f) \]

\[ \alpha: \text{Rolloff Factor} \]

\[ \uparrow \alpha : \uparrow \text{Bandwidth leakage} \]

\[ \downarrow \alpha : \downarrow \text{Bandwidth leakage} \]
Pulse Shaping: Raised Cosine

\[ p(t) = \text{sinc}(t/T_b) \frac{\cos(\pi \alpha t/T_b)}{1 - 4 \alpha^2 t^2/T_b^2} \]

**\( \alpha \): Rolloff Factor**

\( \uparrow \alpha : \uparrow \) Bandwidth leakage, \( \downarrow \) Time Support, \( \downarrow \) Sidelobes \( \Rightarrow \downarrow \) ISI if sampling not aligned

\( \downarrow \alpha : \downarrow \) Bandwidth leakage, \( \uparrow \) Time Support, \( \uparrow \) Sidelobes \( \Rightarrow \uparrow \) ISI if sampling not aligned
Wireless Communication System

Bits-to-Symbols Mapper
Modulation (Encoding)

Bits-to-Symbols Mapper
Symbols-to-Bits Mapper

Demodulation (Decoding)
Channel Equalization & Synchronization

Matched Filter

DAC
Pulse Shaping

LPF
BPF
Mixer
PLL
PA

LNA
BPF
LPF
ADC

Mixer

1011010110011001
1011010110011001
Wireless Communication System

TX

Pulse Shaping

RX

Matched Filter
Pulse Shaping and Matched Filtering

1011010110011001

TX

RX

10110101110011001

Pulse Shaping

Matched Filter
Pulse Shaping and Matched Filtering
Pulse Shaping and Matched Filtering

\[ s[n] \Rightarrow x(t) = s(t) * p_T(t) \]
Consider Simple AWGN Channel. 

$v(t)$ is Gaussian noise.
Pulse Shaping and Matched Filtering

\[ s[n] \rightarrow x(t) = s(t) \ast p_T(t) \rightarrow y(t) = x(t) + v(t) \rightarrow \tilde{y}(t) = y(t) \ast p_R(t) \]

\[ \tilde{y}(t) = s(t) \ast p_T(t) \ast p_R(t) + v(t) \ast p_R(t) \]
Pulse Shaping and Matched Filtering

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s[n] \rightarrow x(t) = s(t) \ast p_T(t) \rightarrow y(t) = x(t) + v(t) \rightarrow \tilde{y}(t) = y(t) \ast p_R(t)
\]

\[
\tilde{y}(t) = s(t) \ast p_T(t) \ast p_R(t) + v(t) \ast p_R(t)
\]

**GOAL:** Find \( P_R(f) \) that maximizes the SNR

**SNR is maximized when:** \( p_R(t) = p_T^*(-t) \)

\( p_R(t) \) matches \( p_T(t) \)… hence, the name matched filter!
Pulse Shaping and Matched Filtering

\[ s[n] \rightarrow x(t) = s(t) \ast p_T(t) \rightarrow y(t) = x(t) + v(t) \rightarrow \tilde{y}(t) = y(t) \ast p_R(t) \]

\[ \tilde{y}(t) = s(t) \ast p_T(t) \ast p_R(t) + v(t) \ast p_R(t) \]
Pulse Shaping and Matched Filtering

\[ s[n] \rightarrow x(t) = s(t) * p_T(t) \rightarrow y(t) = x(t) + v(t) \rightarrow \tilde{y}(t) = y(t) * p_R(t) \]

\[ \tilde{y}(t) = s(t) * p_T(t) * p_T^*(-t) + v(t) * p_T^*(-t) \]

\( p(t) \) is the pulse we need i.e. raised cosine.

- Let \( p(t) \) be a raised cosine
- What is \( p_T(t) \)?

\[ P(f) = \mathcal{F}\{p(t)\} = \mathcal{F}\{p_T(t) * p_T^*(-t)\} = P_T(f) \cdot P_T^*(f) = |P_T(f)|^2 = P_{rc}(f) \]

\[ |P_T(f)| = \sqrt{P_{rc}(f)} = P_{srrc}(f) \quad \Rightarrow \text{Use square-root of raised cosine filter} \]
Pulse Shaping and Matched Filtering

\[ s[n] \rightarrow x(t) = s(t) \ast p_T(t) \rightarrow y(t) = x(t) + v(t) \rightarrow \tilde{y}(t) = y(t) \ast p_R(t) \]

\[ \tilde{y}(t) = s(t) \ast p_T(t) \ast p_T^*(-t) + v(t) \ast p_T^*(-t) \]

* \( p(t) \) is the pulse we need i.e. raised cosine.

- Let \( p(t) \) be a raised cosine
- What is \( p_T(t) \)?

Square-root of raised cosine filter: \( P_{srrc}(f) = \sqrt{P_{rc}(f)} \)

\[ p_{srrc}(t) = -\frac{1}{\sqrt{T_b}} \sin \left( (1 - \alpha) \frac{\pi t}{T_b} \right) + \frac{4\alpha t}{T_b} \cos \left( (1 + \alpha) \frac{\pi t}{T_b} \right) \]

\[ \frac{\pi t}{T_b} \left( 1 - \left( \frac{4\alpha t}{T_b} \right)^2 \right) \]
Which of the following pulses requires the most bandwidth?

A. Rectangular Pulse

B. Sinc Pulse

C. Raised Cosine Pulse

D. Square Root of Raised Cosine Pulse
Pace and Difficulty

A. Too Fast & Too Hard
B. Too Fast & Not Too Hard
C. Good Speed & Too Hard
D. Good Speed & Not Too Hard
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- Types Digital Modulation
- QAM: Maximum Likelihood Decoding & BER vs. SNR

This Lecture:
- Pulse Shaping
- Matched Filter
- Multipath Channel
- Channel Estimation & Correction
- Narrowband vs. Wideband Channels
- Channel Equalization
The Channel

1011010110011001

TX

Bits

Bits-to-Symbols Mapper

Modulation (Encoding)

1011010110011001

RX

Symbols-to-Bits Mapper

Demodulation (Decoding)

Channel

\[ x(t) \rightarrow y(t) = x(t) + v(t) \]

Channel adds noise (AWGN)!
The Channel

Channel delays the signal!

\[ x(t) \rightarrow y(t) = x(t - \tau) + v(t) \]
The Channel

Channel attenuates the signal (Pathloss)

\[ P_{RX} = G_{TX} G_{RX} \frac{\lambda^2}{(4\pi d)^2} P_{TX} \]

\[ |h| \propto \frac{\lambda}{d} \]
The Channel

Channel rotates the signal (Adds Phase)

\[ y(t) = h x(t - \tau) + v(t) \]

\[ h \propto \frac{\lambda}{d} e^{j\phi} \]
The Channel

\[ x(t) \rightarrow y(t) = h x(t - \tau) + v(t) \]

\[ x(t) \times e^{-j2\pi f_c t} \rightarrow |h| x(t - \tau) e^{-j2\pi f_c (t - \tau)} \rightarrow x e^{j2\pi f_c t} \rightarrow |h| x(t - \tau) e^{j2\pi f_c \tau} \]

\[ h \propto \frac{\lambda}{d} e^{j\phi} \rightarrow \phi = 2\pi f_c \tau = 2\pi \frac{c}{\lambda c} = 2\pi \frac{d}{\lambda} \rightarrow h \propto \frac{\lambda}{d} e^{j2\pi d / \lambda} \]
The Channel

Channel:
- Adds Noise
- Delays the Signal
- Attenuates the Signal
- Rotates the Phase of the Signal

\[ x(t) \rightarrow y(t) = h x(t - \tau) + v(t) \]

\[ h \propto \frac{\lambda}{d} e^{j2\pi d/\lambda} \]
The Channel

Consider BPSK Modulation.

\[ 0 \rightarrow -1 \]
\[ 1 \rightarrow +1 \]
The Channel

Consider BPSK Modulation.

\[ 0 \rightarrow -1 \]
\[ 1 \rightarrow +1 \]
Consider QAM Modulation

Demodulating correctly requires COHERENCE! i.e., Need to estimate & correct for the channel $h$

CHANNEL EQUALIZATION
Channel Estimation & Correction

Consider BPSK Modulation.

\[ 0 \rightarrow -1 \]
\[ 1 \rightarrow +1 \]

Send Training Sequence (Preamble Bits): Known Bits

\[
\begin{align*}
  x(0) &= 1 & y(0) &= h + v(0) \\
  x(1) &= 1 & y(1) &= h + n(1) \\
  x(2) &= -1 & y(2) &= -h + n(2) \\
  \vdots & & \vdots
\end{align*}
\]

Estimate channel: \( \tilde{h} = \sum_k \frac{y(k)}{x(k)} \)

Correct channel: \( \tilde{x}(t) = \frac{y(t)}{\tilde{h}} \)
The Channel

\[ x(t) \xrightarrow{\text{Channel}} y(t) = h x(t - \tau) + v(t) \]

\[ h \propto \frac{\lambda}{d} e^{j2\pi d/\lambda} \]

Assumes single path!
Multipath Channel

Multipath Propagation: radio signal reflects off objects ground, arriving at destination at slightly different times

\[ y(t) = \alpha_1 e^{\phi_1} x(t - \tau_1) + \alpha_2 e^{\phi_2} x(t - \tau_2) + \alpha_3 e^{\phi_3} x(t - \tau_3) \ldots \]

\[ y(t) = \sum_k \alpha_k e^{\phi_k} x(t - \tau_k) = \sum_k h(\tau_k) x(t - \tau_k) = h(t) * x(t) \]

\( h(t) \) is channel impulse response.
**Multipath Channel**

$h(t)$ is channel impulse response.

$$y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) = h(t) * x(t)$$

**Multi-tap Channel**

ISI: Inter-Symbol-Interference
Symbols arriving along late paths interfere with following symbols.
Multipath Channel

\( h(t) \) is channel impulse response.

\[
y(t) = \sum \limits_k h(\tau_k) x(t - \tau_k) = h(t) \ast x(t)
\]

- First Path
- Second Path
- nth Path

Multi-tap Channel

**ISI**: Inter-Symbol-Interference
Symbols arriving along late paths interfere with following symbols.

Channel Fading
Symbols arriving along different paths sum up destructively

Paths sum with different phases: Constructive/Destructive
Multipath Channel

$h(t)$ is channel impulse response.

$$y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) = h(t) * x(t)$$

Channel Fading: Symbols arriving along different paths sum up destructively

Example 2 paths with distance $d_1 = 1m, d_2 = 1.06m$:

$$h = h_1 + h_2 = \frac{\lambda}{d_1} e^{j 2\pi d_1 / \lambda} + \frac{\lambda}{d_2} e^{j 2\pi d_2 / \lambda}$$

$$= \frac{\lambda}{d_1} e^{j 2\pi d_1 / \lambda} \left(1 + \frac{d_1}{d_2} e^{j 2\pi (d_2 - d_1) / \lambda}\right) \quad \frac{d_1}{d_2} \approx 1$$

if $\frac{d_2 - d_1}{\lambda} \approx \frac{1}{2} \rightarrow h = \frac{\lambda}{d_1} e^{j 2\pi d_1 / \lambda} (1 + e^{j \pi}) = 0$ Destructive Interference!
Multipath Channel

$h(t)$ is channel impulse response.

\[ y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) = h(t) * x(t) \]

Channel Fading: Symbols arriving along different paths sum up destructively

Example 2 paths with distance $d_1 = 1m, d_2 = 1.06m$:

\[ h = h_1 + h_2 = \frac{\lambda}{d_1} e^{j2\pi d_1/\lambda} + \frac{\lambda}{d_2} e^{j2\pi d_2/\lambda} \]

@ $f_1 = 2.5GHz$ ($\lambda = 12 \text{ cm}$): $h = 0.12 e^{j\frac{2\pi}{3}} + 0.113 e^{j\frac{5\pi}{3}} \approx 0.006$

@ $f_2 = 5GHz$ ($\lambda = 6 \text{ cm}$): $h = 0.06 e^{j\frac{5\pi}{3}} + 0.05 e^{j\frac{5\pi}{3}} \approx 0.116$

Frequency Selective Fading

$17 \times (24dB)$
Multipath Channel

$h(t)$ is channel impulse response.

\[ y(t) = \sum_k h(\tau_k) x(t - \tau_k) = h(t) * x(t) \quad \Leftrightarrow \quad H(f)X(f) \]

Multi-tap Channel
Multipath Channel

$h(t)$ is channel impulse response.

\[ y(t) = \sum_k h(\tau_k) x(t - \tau_k) = h(t) * x(t) \quad \Leftrightarrow \quad H(f)X(f) \]

Multi-tap Channel

ISI: Inter-Symbol-Interference
Symbols arriving along late paths interfere with following symbols.

Frequency Selective Fading
Symbols arriving along different paths sum up destructively

Problematic in Wideband Channel!
Narrowband Channel

$h(t)$ is channel impulse response.

\[ y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) = h(t) * x(t) \quad \Leftrightarrow \quad H(f)X(f) \]
Narrowband Channel

$h(t)$ is channel impulse response.

\[ y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) = h(t) * x(t) \quad \Leftrightarrow \quad H(f)X(f) \]
Narrowband Channel

\( h(t) \) is channel impulse response.

\[
y(t) = \sum_k h(\tau_k) x(t - \tau_k) = h(t) \ast x(t) \quad \iff \quad H(f)X(f)
\]

\[\text{2MHz}\]
Narrowband Channel

\( h(t) \) is channel impulse response.

\[
y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) = h(t) * x(t) \quad \Leftrightarrow \quad H(f)X(f)
\]

Symbol time: \( T \propto \frac{1}{\text{Bandwidth}} \gg \tau_k \)
Narrowband Channel

$h(t)$ is channel impulse response.

$$y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) = h(t) * x(t) \quad \Leftrightarrow \quad H(f)X(f)$$

Symbol time: $T \propto \frac{1}{\text{Bandwidth}} \gg \tau_k$

![Graph showing Tap Index vs. IH^2](image)

Narrowband Channel

Flat Channel
Narrowband Channel

\( h(t) \) is channel impulse response.

\[
y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) = h(t) * x(t) \quad \Leftrightarrow \quad H(f)X(f)
\]

Symbol time: \( T \propto \frac{1}{\text{Bandwidth}} \gg \tau_k \)

\[
y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) \approx \sum_{k} h(\tau_k) x(t) = \left( \sum_{k} h(\tau_k) \right) x(t) = hx(t)
\]

Narrowband Channel is Approximated by a Single Tap Channel
Narrowband Channel

$h(t)$ is channel impulse response.

\[ y(t) = \sum_{k} h(\tau_k) x(t) = h x(t) \quad \iff \quad h X(f) \]

Symbol time: \( T \propto \frac{1}{\text{Bandwidth}} \gg \tau_k \)

\[ y(t) = \sum_{k} h(\tau_k) x(t - \tau_k) \approx \sum_{k} h(\tau_k) x(t) = \left( \sum_{k} h(\tau_k) \right) x(t) = h x(t) \]

Narrowband Channel is Approximated by a Single Tap Channel
Narrowband vs. Wideband Channel

Narrowband Channel

\[ h x(t) \]

Wideband Channel

\[ h(t) * x(t) \]
Narrowband vs. Wideband Channel

Narrowband Channel

\[ h x(t) \Leftrightarrow h X(f) \]

\[ \approx \text{Single Tap} \]

\[ \approx \text{Flat Channel} \]

Wideband Channel

\[ h(t) \ast x(t) \Leftrightarrow H(f) X(f) \]

\[ \approx \text{Multi-tap Channel (ISI)} \]

\[ \approx \text{Frequency Selective Channel} \]

Need to correct for ISI to be able to decode correctly!
Narrowband Channels do not have multipath

Yes

No
Inter-Symbol-Interference

Sources of ISI:

• Multi-tap Channel

\[ y(t) = h(t) \ast x(t) \]

• Pulse Shaping

\[ y(t) = h(t) \ast p(t) \ast s(t) \]

• Other hardware filters

How to deal with ISI?

Channel Equalization
Channel Equalization

Estimating \( h(t) \)

Correcting for \( h(t) \)

\[
y(t) = h(t) \ast x(t)
\]

\[
\hat{x}(t) = f(t) \ast y(t) = f(t) \ast h(t) \ast x(t)
\]

Ideally, \( f(t) = h^{-1}(t) \)

\[
\hat{x}(t) = h^{-1}(t) \ast h(t) \ast x(t) = \delta(t) \ast x(t) = x(t)
\]
Estimating $h(t)$

$h(t)$ is a multi-tap channel

Can Estimate $h(t)$ as an FIR Filter:

- Limited number of paths
- Longer path power decays quickly
- Estimate as a filter with L taps
- Send a training sequence!
Estimating $h(t)$

Send a training sequence of length $N$: $t[0] \ldots t[N - 1]$

$$y[n] = \sum_{l=0}^{L} h[l] t[n - l] + v[n]$$

We will use Least Squares Estimator:

$$\{\hat{h}[0], \hat{h}[1], \ldots, \hat{h}[L]\} = \text{argmin}_{h[0], h[1], \ldots h[L]} \sum_{n=L}^{N-1} \left| y[n] - \sum_{l=0}^{L} h[l] t[n - l] \right|^2$$
Estimating $h(t)$

Send a training sequence of length N:

$$y[n] = \sum_{l=0}^{L} h[l] t[n - l] + v[n]$$

$$\begin{bmatrix} y[L] \\ y[L + 1] \\ \vdots \\ y[N - 1] \end{bmatrix} = \begin{bmatrix} t[L] & \cdots & t[0] \\ t[L + 1] & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ t[N - 1] & \cdots & t[N - 1 - L] \end{bmatrix} \times \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[L] \end{bmatrix} + \begin{bmatrix} v[0] \\ v[1] \\ \vdots \\ v[L] \end{bmatrix}$$

$$y = Ah + v$$
Estimating $h(t)$

Send a training sequence of length $N$:

$$y[n] = \sum_{l=0}^{L} h[l]t[n - l] + v[n]$$

$$y = Ah + v$$

To solve for $h$, we need: $N - L \geq L + 1 \quad \rightarrow \quad N \geq 2L + 1$

Least Squares Solution: minimize $\|y - Ah\|^2$

$$\hat{h} = (A^\dagger A)^{-1} A^\dagger y$$

$A^\dagger$: Conjugate Transpose
Channel Equalization

- Estimating \( h(t) \)
- Correcting for \( h(t) \)

\[
y(t) = h(t) \ast x(t)
\]

\[
\hat{x}(t) = f(t) \ast y(t) = f(t) \ast h(t) \ast x(t)
\]

Ideally, \( f(t) = h^{-1}(t) \)

\[
\hat{x}(t) = h^{-1}(t) \ast h(t) \ast x(t) = \delta(t) \ast x(t) = x(t)
\]
Correcting for $h(t)$

$$y[n] = \sum_{l=0}^{L} h[l]x[n - l] + v[n]$$

Need to find an inverse filter $f$, such that:

$$\sum_{l=0}^{L'} f[l] \hat{h}[n - l] = \delta[n - d']$$

- $h$ is FIR, $h^{-1}$ is IIR $\rightarrow$ Hard to satisfy exactly.
- $d'$ is equalization delay
- Try different $d$ to find the best equalizer
Correcting for $h(t)$

$$y[n] = \sum_{l=0}^{L} h[l]x[n-l] + v[n]$$

Need to find an inverse filter $f$, such that:

$$\sum_{l=0}^{L'} f[l]\hat{h}[n-l] = \delta[n-d'] \quad L' \geq L$$

$$\begin{bmatrix}
\hat{h}[0] & 0 & \ldots & 0 \\
\hat{h}[1] & \hat{h}[0] & & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\hat{h}[L] & 0 & \hat{h}[L] & \ldots & 0 \\
\end{bmatrix} \times \begin{bmatrix}
f[0] \\
f[1] \\
\vdots \\
f[L']
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1 \\
0
\end{bmatrix}$$

$$\hat{H}f = \delta_{d'}$$

minimize $\|\hat{H}f - \delta_{d'}\|^2$

$$\hat{f} = (\hat{H}^+\hat{H})^{-1}\hat{H}^+\delta_{d'}$$
Channel Equalization

- Estimating $h(t)$
- Correcting for $h(t)$

$$y(t) = h(t) \ast x(t)$$
$$\hat{x}(t) = f(t) \ast y(t) = f(t) \ast h(t) \ast x(t)$$

Ideally, $f(t) = h^{-1}(t)$

$$\hat{x}(t) = h^{-1}(t) \ast h(t) \ast x(t) = \delta(t) \ast x(t) = x(t)$$
Channel Equalization

✔ Estimating $h(t)$

Solve Least Squares

$$\hat{h} = (A^+A)^{-1}A^+y$$

✔ Correcting for $h(t)$

Solve Least Squares

$$\hat{f} = (\hat{H}^+\hat{H})^{-1}\hat{H}^+\delta_d$$

Apply $f(t)$ to $y(t)$

Need to Solve Least Squares Twice!
Computationally Intensive!
Channel Equalization

Direct Least Squares
Skip Estimating h altogether

• Send a training sequence of length N:

\[ y[n] = \sum_{l=0}^{L} h[l]t[n - l] + v[n] \]

• Find inverse filter f such that:

\[ \sum_{l=0}^{L'} f[l]y[n - l] = t[n - d] \rightarrow t[n] = \sum_{l=0}^{L'} f[l]y[n + d - l] \]

\[ t = Yf \quad \rightarrow \quad \text{Least Squares: } \hat{f} = (Y^\dagger Y)^{-1}Y^\dagger t \]
Channel Equalization

- Linear
  - LLSE (Linear Least Squares Error)
  - MMSE (Minimum Mean Squares Error)
  - ZF (Zero Forcing)
- Non-Linear
  - DFE (Decision Feedback)
  - MLSE (Maximum Likelihood Sequence Estimation)

Lower Complexity

Lower Noise

Many more tradeoffs:
ISI vs Noise vs Computation vs Convergence
**List of Mathematical Terms That Appeared in the Lecture**

- \( x(t) \): Transmitted Signal
- \( v(t) \): Additive Gaussian Noise
- \( y(t) \): Received Signal
- \( b[n] \): Bit stream
- \( s[n] \): Symbols
- \( n \): Symbol index at TX
- \( p(t) \): Pulse Shape
- \( P(f) \): Frequency Spectrum of pulse
- \( r[n] \): Sampled values at the receiver.
- \( R_b \): Symbol Rate
- \( \Pi(\ ) \): Rectangle Function
- \( \text{sinc}(\ ) \): Sinc Function
- \( \alpha \): Roll-off Factor of Raised Cosine
- \( (\ )^* \): Complex Conjugate
- \(| \ | \): Magnitude
- \( \mathcal{F}\{\ }\): Fourier Transform
- \( p_T(t) \): Transmitter Pulse
- \( p_R(t) \): Receiver Pulse
- \( P_T(f) \): Spectrum of TX pulse
- \( P_R(f) \): Spectrum of RX pulse
- \( P_{rc}(f) \): Spectrum of raised cosine pulse
- \( P_{srcc}(f) \): Spectrum of square root raised cosine.
- \( \tilde{y}(t) \): Received Signal after Matched Filter
- \( \tau \): Time delay of the signal
- \( h \): Single Tap Channel Coefficient.
- \( \tilde{h} \): Estimate of the Channel Coefficient.
- \( \tau_k \): Time delay of the \( k^{th} \) propagation path
- \( \alpha_k \): Attenuation of the \( k^{th} \) propagation path
- \( \phi_k \): Phase of the \( k^{th} \) propagation path
- \( h(t) \): Multi-Tap Channel Impulse Response
- \( H(f) \): Frequency Response of the Channel
- \( X(f) \): Frequency Spectrum of \( x(t) \)
- \( V(f) \): Frequency Spectrum of noise
- \( Y(f) \): Frequency Spectrum of received signal
- \( h[l] \): Coefficient of the \( l^{th} \) channel tap
- \( \tilde{h}[l] \): Estimate of the channel coefficients
- \( T \) or \( T_b \): Symbol time
- \( * \): Convolution
- \( \lambda \): Wavelength of the signal.
- \( d \): Distance between TX and RX
- \( f_c \): Carrier Frequency
- \( s(t) \): Modulated Symbols
- \( h^{-1}(t) \): Inverse Channel Response
- \( f(t) \): Channel Equalization Filter
- \( \hat{x}(t) \): Equalized Signal
- \( \delta(t) \): Impulse Function
- \( y \): Vector of \( y[n] \) samples
- \( h \): Vector of \( h[l] \) coefficients
- \( A \): Matrix of \( t[n] \) training samples
- \( v \): Vector of \( v[n] \) noise samples
- \( f \): Vector of \( f[l] \) coefficients
- \( \hat{f} \): Estimate of \( f \)
- \( \hat{h} \): Estimate of \( h \)
- \( \hat{f} \): Estimate of \( f \)
- \( \hat{H} \): Toeplitz matrix of \( h \)
- \( t \): Vector of \( t[n] \) training samples
- \( Y \): Matrix of \( y[n] \) received samples
- \( d' \): Equalization delay
- \( (\ )^{-1} \): Matrix Inverse
- \( (\ )^\dagger \): Conjugate Transpose of Matrix
- \( L \): Number of channel taps
- \( L' \): Number of Equalization filter taps
- \( y[n] \): Sampled received signal
- \( t[n] \): Training Sequence
- \( N \): Length of the Training Sequence
- \( f[l] \): Equalization Filter coefficients